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Computation of damped nonlinear normal modes with internal resonances: a boundary value approach

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Summary. This paper considers the computation of nonlinear normal modes (NNMs) defined as two-dimensional manifolds in phase space. Because existing methods use explicit manifold parameterization, NNM computation in the presence of internal resonances requires multiple pairs of constraint coordinates. This paper investigates an alternative method for which the manifold is computed using successive boundary value problems.

Introduction

Pioneered in the 60s by Rosenberg, nonlinear normal modes (NNMs) provide a rigorous theoretical framework for analyzing the dynamics of nonlinear systems. Initially defined as families of synchronous periodic oscillations of the autonomous conservative system, NNMs were generalized to damped systems by Shaw and Pierre. Based on geometric arguments and inspired by the center manifold theory, they defined an NNM as a two-dimensional invariant manifold in phase space [1]. Using a single pair of state variables for manifold parameterization (a displacement and a velocity), a set of partial differential equations (PDEs) was derived. Those PDEs globally describe the manifold's geometry in terms of the remaining state-space variables functionally related to the chosen master pair. The first attempt to numerically solve these PDEs and to compute NNMs as invariant manifolds is that of Pesheck et al. [2]. PDEs were written in modal space and solved using a Galerkin projection. In recent contributions, Touzé and co-workers [3] solved the same PDEs using finite differences whereas Renson and Kerschen [4] used a specific finite element method in configuration space.

In the presence of an internal resonance, a nonlinear coupling between two NNMs exists and the invariant manifold starts to fold. The manifold presents a complex structure embedded in the full phase space. The chosen parameterization becomes inappropriate and fails to further describe the invariant surface. To circumvent this issue, Shaw and co-workers introduced the concept of multi-modal NNMs where the invariant manifold is described by multiple pairs of master variables [5]. While effective, this method still assumes an explicit and global description of the NNM which does not completely solve the intrinsic parameterization issue. In this paper, an alternative method for NNM calculation is explored. Originally proposed by Doedel [7] in the general context of two-dimensional (un)stable invariant manifold calculations, the method locally grows the manifold using successive boundary value problems (BVPs). This allows to compute the NNM beyond any parameterization limit and therefore in the presence of nonlinear modal interactions.

Formulation of the boundary value problems

Using the general first-order form of the equation of motion, $\dot{\mathbf{z}} = \mathbf{g}(\mathbf{z})$, a trajectory on the invariant manifold (i.e., the NNM) is defined as

$$\mathbf{z}'(t) = T\mathbf{g}(\mathbf{z}(t)), \tag{1}$$

$$\mathbf{z}(0) = \mathbf{z_0} + r_0(\cos(\theta)\boldsymbol{\psi}_1 + \sin(\theta)\boldsymbol{\psi}_2), \tag{2}$$

where $(.)'$ denotes the first derivative with respect to the normalized time $t \in [0, 1]$, r_0 is a small parameter, and T is the final time. Equations (1)-(2) parameterize using θ a family of trajectories that start on a small ellipse around the equilibrium point $\mathbf{z_0}$. If the vectors $\boldsymbol{\psi}_1$ and $\boldsymbol{\psi}_2$ are chosen so as to define, at the equilibrium point $\mathbf{z_0}$, the tangent space of a NNM, the trajectories will describe this NNM.

The computation strategy proceeds in two steps. First, a trajectory on the manifold is grown using continuation with T as free parameter and θ arbitrarily fixed at $\theta = \theta_0$ [7]. For a stable (resp., unstable) system, $T < 0$ (resp., $T > 0$), and the computation of the first trajectory is similar to backward time integration. In this problem, Equations (1) and (2) are solved for $\mathbf{Z}$ which groups all the points that discretize the trajectory. During the continuation process, a function measuring the arclength of the trajectory as

$$L = \int_0^1 T \left\|\mathbf{g}(\mathbf{z})\right\| ds. \tag{3}$$

is monitored by the algorithm, and the continuation is stopped when the trajectory reaches a user-defined length $\bar{L}$. Equation (2) together with the condition $L = \bar{L}$ define a BVP that constrains the initial and final conditions of the trajectory. A second continuation step, with θ as parameter, is then performed. All the points that discretize the trajectory are continuously varied to cover the complete invariant manifold. For this last step, T is free to vary. Other stopping criteria than the arclength of the trajectory can be employed. We refer to [7] for additional details about the original method.

The different continuation steps are performed with the software AUTO [6] which solves BVPs using the method of orthogonal collocation with piecewise polynomials. The recourse to the BVP approach has the advantage of being compatible with the subsequent continuation steps. The latter are important because trajectories can be very sensitive with respect to the parameter θ. A continuation approach which controls the step size in θ with respect to the variation of all the trajectory (i.e., $\mathbf{Z}$) guarantees a nice covering of the invariant surface. Moreover, stable manifolds become unstable in backward time, and the recourse to BVP approach is thus arguably more robust than direct time integration.

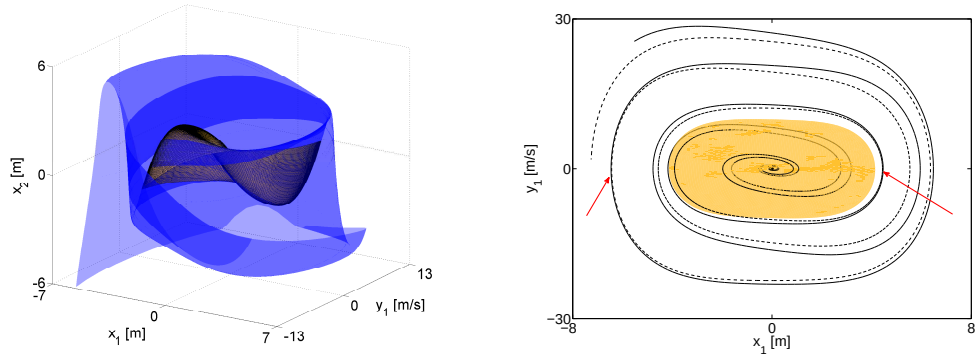
The method is applied to the 2DOF system

$$\begin{aligned} \ddot{x}_1 + 0.3(\dot{x}_1 - \dot{x}_2) + (2x_1 - x_2) + 0.5\,x_1^3 &= 0, \\ \ddot{x}_2 - 0.3(\dot{x}_1 - 2\dot{x}_2) + (2x_2 - x_1) &= 0. \end{aligned} \tag{4}$$

Figure 1(a) presents a three-dimensional view of the second NNM of the system computed with the present approach. The invariant surface is displayed in blue. It is covered with a collection of 2012 trajectories. For comparison, the solution computed with a finite-element-based (FE) algorithm developed in [4] for solving the manifold's governing PDEs (obtained with an explicit manifold parameterization) is also display in orange. Around the origin, both methods agree very well. However, for larger motion amplitudes, the master coordinates (x_1, y_1) employed by the FE method are not adequate anymore to represent the manifold. This is illustrated in Figure 1(b) which presents a projection of two trajectories computed using the BVP approach. The trajectories are getting very close to each other in two specific regions pointed out by arrows. The invariant surface becomes almost vertical, which limits its explicit representation using the pair (x_1, y_1). Conversely, the results of the BVP approach captured the manifold without limitation.

Conclusions

In this paper, an alternative approach for damped NNM computation is investigated. This BVP approach does not assume any explicit parameterization and provides a means to calculate invariant manifolds with complex topologies and thus, for instance, in the presence of internal resonances. A distinctive advantage of the present approach is to provide a direct access to the dynamics on the NNM, and, in turn, to the amplitude-frequency dependence of the NNM.



(a) (b)

Figure 1: Second NNM of the 2DOF system. (a) The present approach (in blue) is compared to the results of [4] in orange. (b) top view in the master coordinates' plane of the FEM. Solid and dashed lines are used to distinguish two trajectories computed using the BVP formulation.

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